

Appendix: Macroeconomic Drivers of Bond and Equity Risks

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This document is a supplemental appendix to Campbell, Pflueger, and Viceira (2018). The contents of this appendix are as follows:

1. Section A provides additional empirical results in support of the assumed link between consumption and the output gap.
2. Section B derives the log-linear expansion for habit around the steady-state.
3. Section C provides details for the model solution for both macroeconomic dynamics and asset pricing moments.
4. Section D describes in detail the implementation of the numerical model solution.
5. Section E provides details for the econometric methodology. Among other details, it describes how we estimate orthogonalized macroeconomic impulse responses. It also describes the regressions of model stock and bond returns onto estimated VAR(1) innovations shown in Table 7, Panel A in the main paper.

A Additional empirical results

Figure A.1 shows that equation (1) in the main paper is a strikingly close description of consumption and output gap data. We regress stochastically detrended consumption onto the output gap:

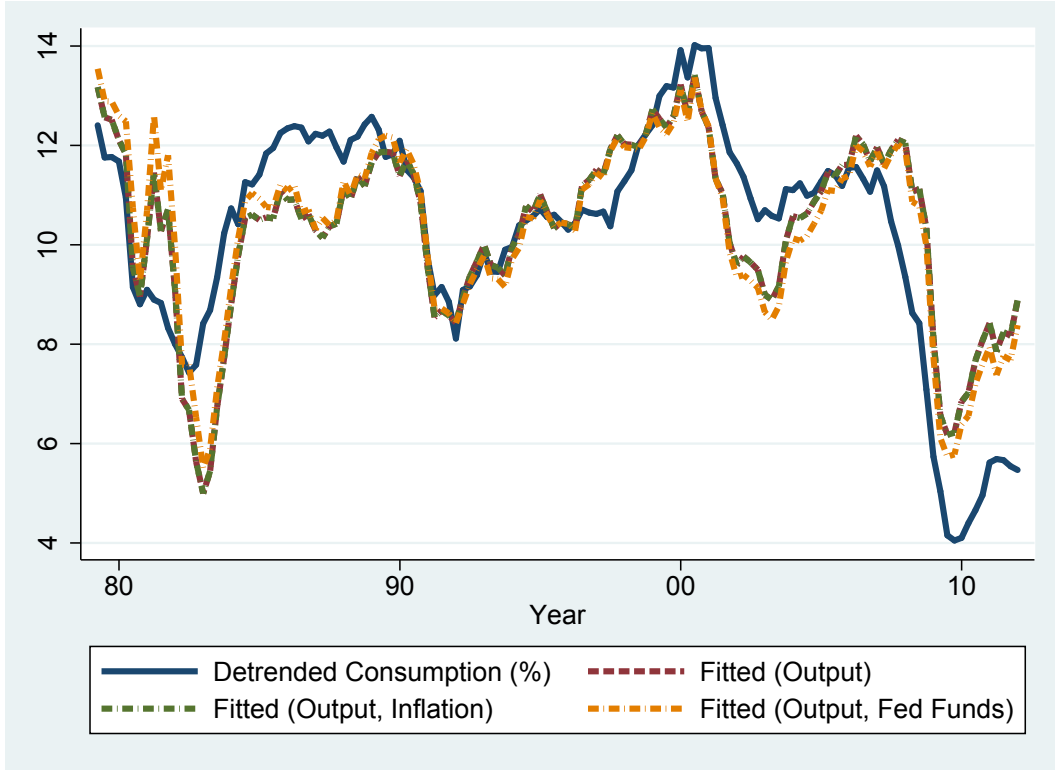
$$\hat{c}_t = b_0 + b_x x_t + \varepsilon_t. \tag{A.1}$$

We set the smoothing parameter to $\phi = 0.93$ (half-life 2.4 years) to maximize the correlation between \hat{c}_t and the output gap. Figure A.1 shows that \hat{c}_t and its fitted value from (A.1) track each other remarkably closely through different macroeconomic regimes, with a correlation of 77%. The estimated slope coefficient b_x is statistically indistinguishable from one.

To test whether the functional form of equation (1) in the main paper is actually a good description of the data, we add inflation and the nominal Federal Funds rate as additional controls to (A.1). We find that neither of these variables enter in a statistically or economically significant manner.

Specifically, our regression results are as follows. “Fitted (Output)” is the fitted value from the quarterly regression $\hat{c}_t = \underset{(0.27)}{11.73} + \underset{(0.17)}{0.88}x_t + \varepsilon_t$; “Fitted (Output, Inflation)” is the fitted value from the quarterly regression $\hat{c}_t = \underset{(0.45)}{11.72} + \underset{(0.17)}{0.88}x_t + \underset{(0.12)}{0.00}\pi_t + \varepsilon_t$; “Fitted (Output, Fed Funds)” is the fitted value from the quarterly regression $\hat{c}_t = \underset{(0.54)}{11.16} + \underset{(0.14)}{0.86}x_t + \underset{(0.08)}{0.09}i_t + \varepsilon_t$. Newey-West standard errors with four lags are shown in parentheses and all regressions are for the full sample 1979Q3-2011Q4.

Figure A.1: Stochastically Detrended Consumption vs. Output Gap



This figure plots the time series of stochastically exponentially detrended consumption $\hat{c}_t = c_t - (1 - \phi)[c_{t-1} + \phi c_{t-2} + \dots]$. The smoothing parameter $\phi = 0.93$, corresponding to a half-life of 2.4 years, is chosen to maximize the univariate correlation between the log real output gap and exponentially detrended consumption. “Fitted(Output)” is the fitted value from the quarterly regression $\hat{c}_t = 11.73 + 0.88 x_t + \varepsilon_t$; “Fitted(Output, Inflation)” is the fitted value from the quarterly regression $\hat{c}_t = 11.72 + 0.88 x_t + 0.00 \pi_t + \varepsilon_t$; “Fitted(Output, Fed Funds)” is the fitted value from the quarterly regression $\hat{c}_t = 11.16 + 0.86 x_t + 0.09 i_t + \varepsilon_t$. All regressions use the full sample 1979Q3-2011Q4 and show Newey-West standard errors with four lags in parentheses. Data for consumption, the output gap, inflation, and the Federal Funds rate are the same as in Table 2.

B Loglinear habit dynamics around steady state

This section derives the loglinear dynamics of the habit stock around the steady-state and compares them to CC. We start by showing that a loglinear approximation around the nonstochastic steady-state allows us to express log habit dynamics as a linear function in c_t, c_{t-1}, \dots

Approximating

$$\hat{s}_t = \left(1 - \frac{1}{S}\right) (h_t - c_t - h), \quad (\text{B.1})$$

we get loglinear approximate dynamics for log habit h_t (ignoring constants):

$$h_{t+1} \approx \theta_0 h_t + (1 - \theta_0) c_t + \frac{\theta_1}{(1 - \frac{1}{S})} x_t + \frac{\theta_2}{(1 - \frac{1}{S})} x_{t-1}, \quad (\text{B.2})$$

$$= (1 - \theta_0) \sum_{j=0}^{\infty} \theta_0^j c_{t-j} - \frac{\theta_1}{(\frac{1}{S} - 1)} \left(c_t - (1 - \phi) \sum_{j=1}^{\infty} \phi^j c_{t-j} \right) \quad (\text{B.3})$$

$$- \frac{\theta_2}{(\frac{1}{S} - 1)} \left(c_{t-1} - (1 - \phi) \sum_{j=1}^{\infty} \phi^j c_{t-1-j} \right). \quad (\text{B.4})$$

When $\theta_1 = \theta_2 = 0$ (the CC case), expression (B.4) shows that log habit is approximately an exponentially-weighted moving average of lagged log consumption. As θ_1 and θ_2 increase, a_0 and a_1 decrease while the weight on longer lags increases. Positive values for θ_1 and θ_2 , as in our calibration, therefore increase the dependence of habit on the most recent lags of consumption relative to CC. If $\phi < \theta_0$, as in our calibration, $a_j / ((1 - \theta_0)\theta_0^j) \rightarrow 1$ as $j \rightarrow \infty$. Intuitively, log habit loads onto long lags of consumption exactly as in CC.

C Details: Model solution

C.1 Solving for macroeconomic dynamics

C.1.1 Writing the problem in matrix form

For the numerical solution, it is convenient to first write the problem in matrix form and orthogonalize the shocks. Recall that the standard deviations and correlations of the model shocks in equations (15)-(17) in the main paper are denoted by $\sigma_\pi, \sigma_i, \sigma_*$ and $\rho_{\pi i}, \rho_{\pi*}, \rho_{i*}$. We define a vector of orthogonal shocks $u_t = [u_{2t}, u_{3t}, u_{*t}]'$ such that $u_{*t} = v_{*t}$ and such that u_t spans the vector of model shocks $v_t = [v_{\pi t}, v_{it}, v_{*t}]'$.

We define the orthogonal shocks u_t via:

$$v_t = M_{temp} u_t, \quad (\text{C.1})$$

where the invertible transformation matrix M_{temp} is given by:

$$M_{temp} = \begin{bmatrix} 1 & 0 & \frac{\rho_{\pi*}\sigma_\pi}{\sigma_*} \\ \frac{\sigma_i(\rho_{\pi i} - \rho_{\pi*}\rho_{i*})}{\sigma_\pi(1 - \rho_{\pi*}^2)} & 1 & \frac{\sigma_i\rho_{i*}}{\sigma_*} \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{C.2})$$

We denote the variance-covariance matrix of the orthogonalized shocks u_t by Σ_u .

We can solve for it in terms of the model parameters and M_{temp} :

$$\Sigma_u = E_t [u'_t u_t] \quad (C.3)$$

$$= M_{temp}^{-1} \begin{bmatrix} \sigma_\pi^2 & \sigma_\pi \sigma_i \rho_{\pi i} & \sigma_\pi \sigma_* \rho_{\pi*} \\ \sigma_i \sigma_\pi \rho_{\pi i} & \sigma_i^2 & \sigma_i \sigma_* \rho_{i*} \\ \sigma_\pi \sigma_* \rho_{\pi*} & \sigma_i \sigma_* \rho_{i*} & \sigma_*^2 \end{bmatrix} M_{temp}^{-1'} \quad (C.4)$$

where the matrix $\begin{bmatrix} \sigma_\pi^2 & \sigma_\pi \sigma_i \rho_{\pi i} & \sigma_\pi \sigma_* \rho_{\pi*} \\ \sigma_i \sigma_\pi \rho_{\pi i} & \sigma_i^2 & \sigma_i \sigma_* \rho_{i*} \\ \sigma_\pi \sigma_* \rho_{\pi*} & \sigma_i \sigma_* \rho_{i*} & \sigma_*^2 \end{bmatrix}$ is the variance-covariance matrix of the vector v_t .

We can write the macroeconomic dynamics in matrix form as:

$$0 = FE_t \hat{Y}_{t+1} + G\hat{Y}_t + H\hat{Y}_{t-1} + Mu_t, \quad (C.5)$$

where

$$F = \begin{bmatrix} f^x & \psi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (C.6)$$

$$G = \begin{bmatrix} -1 & 0 & -\psi \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (C.7)$$

$$H = \begin{bmatrix} \rho^x & 0 & 0 \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}, \quad (C.8)$$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{\rho_{\pi*} \sigma_\pi}{\sigma_*} \\ m_{31} & 1 & m_{33} \end{bmatrix}. \quad (C.9)$$

where

$$m_{31} = \frac{\sigma_i (\rho_{\pi i} - \rho_{\pi*} \rho_{i*})}{\sigma_\pi (1 - \rho_{\pi*}^2)} \quad (C.10)$$

and

$$m_{33} = \frac{\sigma_i \rho_{i*}}{\sigma_*}. \quad (C.11)$$

C.1.2 Method of generalized eigenvalues

We consider only solutions of the form:

$$\hat{Y}_t = P\hat{Y}_{t-1} + Qu_t. \quad (C.12)$$

Each solution (C.12) corresponds to a unique solution of the form (20) in the main paper, where the matrices Σ and Q have the following one-to-one mapping:

$$\Sigma = QM_{temp}^{-1}. \quad (C.13)$$

Additional solutions, such as solutions depending on two lags of state variables, may exist, but we do not consider them. A solution of the form (C.12) must satisfy the quadratic matrix equation:

$$FP^2 + GP + H = 0. \quad (\text{C.14})$$

To solve for the matrix P , we proceed in two steps. First, we determine all possible matrices P . Second, we pick one out of several solutions. For the first step, we follow Uhlig (1999) and solve for the generalized eigenvectors and eigenvalues of Ξ with respect to Δ , where:

$$\Xi = \begin{bmatrix} -G & -H \\ I_3 & 0_3 \end{bmatrix}, \quad (\text{C.15})$$

$$\Delta = \begin{bmatrix} F & 0_3 \\ 0_3 & I_3 \end{bmatrix}. \quad (\text{C.16})$$

A generalized eigenvector \mathbf{z} is a non-zero $[6 \times 1]$ vector with associated eigenvalue b , such that

$$\Xi \mathbf{z} = b \cdot \Delta \mathbf{z}. \quad (\text{C.17})$$

We solve for b and \mathbf{z} using the MATLAB function $\text{eig}(Xi, Delta)$. Since Ξ and Δ are $[6 \times 6]$, there are in general six generalized eigenvectors, $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_6$, with associated generalized eigenvalues b_1, b_2, \dots, b_6 . Uhlig (1999) proves that there exist $[3 \times 1]$ vectors $\tilde{\mathbf{z}}_1, \dots, \tilde{\mathbf{z}}_6$, such that the generalized eigenvectors take the form

$$\mathbf{z}'_i = [b_i \tilde{\mathbf{z}}'_i, \tilde{\mathbf{z}}'_i]. \quad (\text{C.18})$$

Let $\{i_1, i_2, i_3\} \subset \{1, 2, 3, 4, 5, 6\}$ denote a selection of three eigenvectors with associated eigenvalues. We define the matrices

$$B_{i_1, i_2, i_3} = \text{diag}(b_{i_1}, b_{i_2}, b_{i_3}), \quad (\text{C.19})$$

$$\Omega_{i_1, i_2, i_3} = [\tilde{\mathbf{z}}_{i_1}, \tilde{\mathbf{z}}_{i_2}, \tilde{\mathbf{z}}_{i_3}]. \quad (\text{C.20})$$

Uhlig (1999) proves that any matrix P_{i_1, i_2, i_3} of the following form is a solution to (C.14):

$$P_{i_1, i_2, i_3} = \Omega_{i_1, i_2, i_3} B_{i_1, i_2, i_3} \Omega_{i_1, i_2, i_3}^{-1}. \quad (\text{C.21})$$

The expression for P_{i_1, i_2, i_3} makes clear that b_{i_1}, b_{i_2} , and b_{i_3} are the eigenvalues of P_{i_1, i_2, i_3} . Since we have to pick three generalized eigenvalues out of six, there are multiple matrices P solving the matrix quadratic equation (C.14).

For any solution P_{i_1, i_2, i_3} , the corresponding matrix Q_{i_1, i_2, i_3} is given by:

$$Q_{i_1, i_2, i_3} = -[FP_{i_1, i_2, i_3} + G]^{-1}M. \quad (\text{C.22})$$

To write the equilibrium dynamics in the form (20) in the main paper, one can obtain the matrix Σ by plugging Q_{i_1, i_2, i_3} into (C.13).

Because equations (15) and (16) in the main paper are already in the form of a VAR(1), rows 2 and 3 of P_{i_1, i_2, i_3} simply list the lag parameters of those equations. Because equation (13) in the main paper has a forward-looking term, the first row of P is where the different solutions P_{i_1, i_2, i_3} differ.

So far, the solution method has been the same irrespective of the number of generalized eigenvalues with $\|\lambda\| > 1$ (explosive eigenvalues). However, the procedure for selecting one solution out of several depends on the number of explosive eigenvalues. Each solution for P and Q corresponds to a different set of three eigenvalues. We discard equilibria where at least one eigenvalue of P is explosive or where eigenvalues do not occur in complex conjugate pairs. We also discard equilibria where a real eigenvalue is less than -0.2 , and where at least one of the model impulse responses shown in Figures 3, 4, and 5 switches sign within the first four quarters. If at a given set of parameter values we only have one equilibrium after imposing these requirements, we pick that equilibrium. For any set of parameters where we still have multiple equilibria, we pick the equilibrium with the smallest simulated method of moments (SMM) objective function, with the weight on the Federal Funds rate-output gap correlation set to zero.

C.2 Asset pricing recursion

Before deriving the asset pricing recursions, we start by deriving some expressions that will be useful repeatedly. We use e_i to denote a row vector with 1 in position i and zeros elsewhere.

We can use the link between stochastically detrended consumption and the output gap (equation (1) in the main paper) to express consumption in terms of the current and lagged output gap:

$$c_t = g + c_{t-1} + x_t - \phi x_{t-1}. \quad (\text{C.23})$$

It follows that log consumption growth equals:

$$c_{t+1} - c_t = g + x_{t+1} - \phi x_t. \quad (\text{C.24})$$

We will substitute (C.24) repeatedly into the stochastic discount factor.

The matrix

$$Q_M = e_1 Q \quad (\text{C.25})$$

denotes the loading of consumption innovations onto the vector of shocks u_t , where e_1 is a basis vector with a one in the first position and zeros everywhere else. The volatility of consumption surprises equals:

$$\sigma_c^2 = Q_M \Sigma_u Q_M'. \quad (\text{C.26})$$

To simplify notation, we define \hat{s}_t as the log deviation of surplus consumption from its steady state. The dynamics of \hat{s}_t are:

$$\hat{s}_t = s_t - \bar{s}, \quad (\text{C.27})$$

$$\hat{s}_t = \theta_0 \hat{s}_{t-1} + \theta_1 x_{t-1} + \theta_2 x_{t-2} + \lambda(\hat{s}_{t-1}) Q_M u_t, \quad (\text{C.28})$$

where with an abuse of notation we write:

$$\lambda(\hat{s}_t) = \lambda_0 \sqrt{1 - 2\hat{s}_t} - 1, \hat{s}_t \leq s_{max} - \bar{s}, \quad (\text{C.29})$$

$$\lambda(\hat{s}_t) = 0, \hat{s}_t \geq s_{max} - \bar{s}. \quad (\text{C.30})$$

The steady-state surplus consumption sensitivity equals:

$$\lambda_0 = \frac{1}{\bar{S}}. \quad (\text{C.31})$$

The steady-state output gap is normalized to zero. The steady-state real short-term interest rate at $x_t = 0$ and $s_t = \bar{s}$ is then the same as in CC:

$$\bar{r} = \gamma g - \frac{1}{2} \gamma^2 \sigma_c^2 / \bar{S}^2 - \log(\beta). \quad (\text{C.32})$$

In our calculations of asset prices, we repeatedly use the following expression for the expected growth in the log SDF:

$$E_t [m_{t+1}] = \log(\beta) - \gamma g + \gamma \hat{s}_t + \gamma \phi x_t - \gamma \mathbb{E}_t \hat{s}_{t+1} - \gamma \mathbb{E}_t x_{t+1} \quad (\text{C.33})$$

$$= -r_t - \frac{\gamma}{2} (1 - \theta_0) (1 - 2\hat{s}_t), \quad (\text{C.34})$$

which follows from the asset pricing Euler equation for the real short rate. We often combine (C.34) with $r_t = \bar{r} + (e_3 - e_2 P) Z_t$.

C.2.1 State space

Our state space for solving for asset prices is five-dimensional: It consists of \tilde{Z}_t , which is a scaled version of \hat{Y}_t , the surplus consumption ratio relative to steady-state \hat{s}_t , and the lagged output gap x_{t-1} .

We next describe the definition of \tilde{Z}_t . To simplify the numerical implementation of the asset pricing recursions, it is convenient to define a scaled state vector. We require that shocks to the scaled state vector are independent standard normal and that the first dimension of the scaled state vector is perfectly correlated with output gap innovations. This rotation facilitates the numerical analysis, because it is easier to integrate over independent random variables. Aligning the first dimension of the scaled state vector with output gap innovations (and hence surplus consumption

innovations) helps, because it allows us to use a finer grid to integrate numerically over this crucial dimension, where asset prices are most nonlinear.

If the scaled state vector equals $\tilde{Z}_t = A\hat{Y}_t$ for some invertible matrix A , the dynamics of \tilde{Z}_t are given by:

$$\tilde{Z}_t = A\hat{Y}_t, \quad (\text{C.35})$$

$$\tilde{Z}_{t+1} = \underbrace{APA^{-1}}_{\tilde{P}}\tilde{Z}_t + \underbrace{AQ}_{\epsilon_{t+1}}u_{t+1}. \quad (\text{C.36})$$

What should be the variance-covariance matrix of ϵ_{t+1} and how does it constrain our choice of A ? Note that the matrix Q has rank two, because the matrix M defined in equation (C.9) has rank two, and Q and M are related via (C.22). We therefore want A such that the vector ϵ_{t+1} has two dimensions distributed as independent standard normals and the third one identically equal to zero. That is, we need A such that

$$\text{Var}_t(\epsilon_{t+1}) = AQ\Sigma_uQ'A', \quad (\text{C.37})$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (\text{C.38})$$

Requiring that the first dimension of ϵ_{t+1} is perfectly correlated with output-gap surprises gives a second constraint for A :

$$e_1A = (\sigma_c)^{-1}e_1. \quad (\text{C.39})$$

Letting A_i denote the i th row of A ($i = 1, 2, 3$), we compute A using the following three steps.

1. We set

$$A_1 = (\sigma_c)^{-1}e_1. \quad (\text{C.40})$$

This ensures that condition (C.39) is satisfied.

2. We use the MATLAB function ‘null’ to compute the null space $\text{null}(A_1Q\Sigma_uQ')$. We define n_2 as the first vector in $\text{null}(A_1Q\Sigma_uQ')$, so by definition we know that $n_2(A_1Q\Sigma_uQ)' = 0$. We then define the second column of A as

$$A_2 = \frac{n_2}{\sqrt{n_2Q\Sigma_uQn_2'}. \quad (\text{C.41})$$

3. We define the vector $n_3 = \text{null}(Q')$, so by definition $n_3Q = 0$. We then define the third row of A as

$$A_3 = n_3. \quad (\text{C.42})$$

It is then straightforward to verify that (C.38) holds for

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}. \quad (\text{C.43})$$

C.2.2 Recursion for zero-coupon consumption claims

We now derive the recursion for zero-coupon consumption claims in terms of state variables \tilde{Z}_t , \hat{s}_t and x_{t-1} . Let P_{nt}^c/C_t denote the price-dividend ratio of a zero-coupon claim on consumption at time $t+n$. The outline of our strategy here is that we first derive an analytic expression for the price-dividend ratio for P_{1t}^c/C_t . For $n \geq 1$ we guess and verify recursively that there exists a function $F_n(\tilde{Z}_t, \hat{s}_t, x_{t-1})$, such that

$$\frac{P_{nt}^c}{C_t} = F_n(\tilde{Z}_t, \hat{s}_t, x_{t-1}). \quad (\text{C.44})$$

We start by deriving the analytic expression for F_1 . The one-period zero coupon price-consumption ratio solves

$$\frac{P_{1,t}^c}{C_t} = \mathbb{E}_t \left[\frac{M_{t+1}C_{t+1}}{C_t} \right] \quad (\text{C.45})$$

Using (C.23) to substitute for consumption growth, we factorize $M_{t+1} \frac{C_{t+1}}{C_t}$:

$$\begin{aligned} M_{t+1} \frac{C_{t+1}}{C_t} &= \beta \exp(-\gamma(\hat{s}_{t+1} - \hat{s}_t) - (\gamma - 1)(c_{t+1} - c_t)) \\ &= \beta \exp(-\gamma(\hat{s}_{t+1} - \hat{s}_t) - (\gamma - 1)(g + x_{t+1} - \phi x_t)) \end{aligned} \quad (\text{C.46})$$

Using the notation $f_n = \log(F_n)$, (C.28) and (C.34) give:

$$\begin{aligned} f_1(\tilde{Z}_t, \hat{s}_t, x_{t-1}) &= \log(\beta) - (\gamma - 1)g + \gamma\hat{s}_t + (\gamma - 1)\phi x_t \\ &\quad - \gamma E_t \hat{s}_{t+1} - (\gamma - 1) E_t x_{t+1} \\ &\quad + \frac{1}{2}(\gamma\lambda(\hat{s}_t) + (\gamma - 1))^2 \sigma_c^2, \\ &= g + e_1[P - \phi I]A^{-1}\tilde{Z}_t + \frac{1}{2}(\gamma\lambda(\hat{s}_t) + (\gamma - 1))^2 \sigma_c^2 \\ &\quad - \bar{r} - (e_3 - e_2 P)A^{-1}\tilde{Z}_t - \frac{\gamma}{2}(1 - \theta_0)(1 - 2\hat{s}_t). \end{aligned} \quad (\text{C.47})$$

Next, we solve for f_n , $n \geq 2$ iteratively. Note that:

$$\frac{P_{nt}^c}{C_t} = \mathbb{E}_t \left[\frac{M_{t+1}C_{t+1}}{C_t} \frac{P_{n-1,t+1}^c}{C_{t+1}} \right] = \mathbb{E}_t \left[\frac{M_{t+1}C_{t+1}}{C_t} F_{n-1}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t) \right] \quad (\text{C.48})$$

This gives the following expression for f_n :

$$f_n(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \log \left[\mathbb{E}_t \left[\exp \left(\log(\beta) - (\gamma - 1)g + \gamma \hat{s}_t + (\gamma - 1)\phi x_t - \gamma \hat{s}_{t+1} - (\gamma - 1)\mathbb{E}_t x_{t+1} - (\gamma - 1)e_1 A^{-1} e'_1 \epsilon_{1,t+1} + f_{n-1}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t) \right) \right] \right]. \quad (\text{C.49})$$

Here, $\epsilon_{1,t+1}$ denotes the first dimension of the shock ϵ_{t+1} . We clarify that the expression in parentheses depends only on the first shock to the scaled state vector, $\epsilon_{1,t+1}$. Finally, we use (C.34) to re-write $f_{n,t}$ as an expectation involving $f_{n-1,t+1}$, the state variables \tilde{Z}_t , \hat{s}_t , and x_{t-1} , and $\epsilon_{1,t+1}$:

$$f_n(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \log \left[\mathbb{E}_t \left[\exp \left(g + e_1 [P - \phi I] A^{-1} \tilde{Z}_t - \bar{r} - (e_3 - e_2 P) A^{-1} \tilde{Z}_t - \frac{\gamma}{2} (1 - \theta_0) (1 - 2\hat{s}_t) - (\gamma(1 + \lambda(\hat{s}_t)) - 1) \sigma_c \epsilon_{1,t+1} + f_{n-1}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t) \right) \right] \right]. \quad (\text{C.50})$$

C.2.3 Recursion for zero-coupon bond prices

We use $P_{n,t}^{\$}$ and $P_{n,t}$ denote the prices of nominal and real n -period zero-coupon bonds. The strategy is to develop analytic expressions for one- and two-period bond prices. We then guess and verify recursively that the prices of real and nominal zero-coupon bonds with maturity $n \geq 2$ can be written in the following form:

$$P_{n,t} = B_n(\tilde{Z}_t, \hat{s}_t, x_{t-1}), \quad (\text{C.51})$$

$$P_{n,t}^{\$} = \exp(-n\pi_{*t}) B_n^{\$}(\tilde{Z}_t, \hat{s}_t, x_{t-1}), \quad (\text{C.52})$$

where $B_n(\tilde{Z}_t, \hat{s}_t, x_{t-1})$ and $B_n^{\$}(\tilde{Z}_t, \hat{s}_t, x_{t-1})$ are functions of the state variables.

As discussed in the main paper, we assume that the short-term nominal interest rate contains no risk premium, so the one-period log nominal interest rate equals $\hat{i}_t = r_t + E_t \pi_{t+1}$. The means of \hat{i}_t and $\hat{\pi}_t$ are normalized to zero, but in order to derive bond prices we need to account for the average level of interest rates. We do this by writing the one-period log nominal interest rate as $\hat{i}_t = \hat{i}_t + \pi_{*t} + \bar{r}$ and the one-period log real interest rate as $r_t = \hat{i}_t - E_t \hat{\pi}_{t+1} + \bar{r}$. One-period bond prices then equal:

$$P_{1,t}^{\$} = \exp(-\hat{i}_t - \pi_{*t} - \bar{r}), \quad (\text{C.53})$$

$$P_{1,t} = \exp(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r}). \quad (\text{C.54})$$

We next solve for longer-term bond prices including risk premia. Substituting in

(C.53) into the bond-pricing recursion gives:

$$P_{2,t}^{\S} = \mathbb{E}_t \left[M_{t+1} P_{1,t+1}^{\S} \exp(-\pi_{*t+1} - \hat{\pi}_{t+1}) \right] \quad (\text{C.55})$$

$$= \mathbb{E}_t \left[M_{t+1} \exp(-\hat{i}_{t+1} - 2\pi_{*t+1} - \hat{\pi}_{t+1} - \bar{r}) \right] \quad (\text{C.56})$$

$$= \beta \mathbb{E}_t \left[\exp(-\gamma(\hat{s}_{t+1} - \hat{s}_t) - \gamma(g + x_{t+1} - \phi x_t) - \hat{i}_{t+1} - 2\pi_{*t+1} - \hat{\pi}_{t+1} - \bar{r}) \right]. \quad (\text{C.57})$$

We can now verify that the two-period nominal bond price takes the form (C.52):

$$\begin{aligned} B_2^{\S}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) &= \exp(\log(\beta) - \gamma g + \gamma \hat{s}_t + \gamma \phi x_t) \\ &\quad \times \exp\left(\mathbb{E}_t\left(-\gamma \hat{s}_{t+1} - \gamma x_{t+1} - \hat{i}_{t+1} - \hat{\pi}_{t+1} - \bar{r}\right)\right) \\ &\quad \times \mathbb{E}_t \left[\exp\left(\left(-\gamma(\lambda(\hat{s}_t) + 1) Q_M - \underbrace{[(e_2 + e_3)Q + 2e_3]}_{v_{\S}}\right) u_{t+1}\right) \right]. \end{aligned} \quad (\text{C.58})$$

Here, we define the vector v_{\S} to simplify notation. The random walk component of inflation π_{*t} does not appear in (C.58), because B_2^{\S} is already scaled by $\exp(-2\pi_{*t})$ by definition (C.52). Taking logs, applying (C.34), and using the definition for the sensitivity function $\lambda(\hat{s}_t)$, we get:

$$\begin{aligned} b_2^{\S} &= -e_3[I + P]A^{-1}\tilde{Z}_t + \frac{1}{2}v_{\S}\Sigma_u v_{\S}' \\ &\quad + \gamma(\lambda(\hat{s}_t) + 1)Q_M\Sigma_u v_{\S}' - 2\bar{r}. \end{aligned} \quad (\text{C.59})$$

We similarly solve for two-period real bond prices in closed form:

$$\begin{aligned} P_{2,t} &= \exp(\log(\beta) - \gamma g + \gamma \hat{s}_t + \gamma \phi x_t) \\ &\quad \times \exp\left(\mathbb{E}_t\left(-\gamma \hat{s}_{t+1} - \gamma x_{t+1} - \hat{i}_{t+1} + \mathbb{E}_{t+1}\hat{\pi}_{t+2} - \bar{r}\right)\right) \\ &\quad \times \mathbb{E}_t \left[\exp\left(\left(-\gamma(\lambda(\hat{s}_t) + 1)Q_M - \underbrace{(e_3 - e_2P)Q}_{v_r}\right) u_{t+1}\right) \right] \end{aligned} \quad (\text{C.60})$$

We define the vector v_r to simplify notation. Taking logs, applying (C.34), and using the definition for $\lambda(\hat{s}_t)$ gives:

$$b_2(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = -(e_3 - e_2P)[I + P]A^{-1}\tilde{Z}_t + \frac{1}{2}v_r\Sigma_u v_r' + \gamma(\lambda(\hat{s}_t) + 1)Q_M\Sigma_u v_r' - 2\bar{r}. \quad (\text{C.61})$$

For $n \geq 3$, we use (C.34) to obtain the following recursion for real bond prices:

$$B_n(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \mathbb{E}_t \left[\exp(\log(\beta) - \gamma g + \gamma \hat{s}_t - \gamma \hat{s}_{t+1} + \gamma \phi x_t - \gamma x_{t+1} + b_{n-1}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t)) \right] \quad (\text{C.62})$$

$$= \mathbb{E}_t \left[\exp \left(-\bar{r} - (e_3 - e_2 P) A^{-1} \tilde{Z}_t - \frac{\gamma}{2} (1 - \theta_0) (1 - 2\hat{s}_t) - \gamma (1 + \lambda(\hat{s}_t)) \sigma_c \epsilon_{1,t+1} + b_{n-1}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t) \right) \right]. \quad (\text{C.63})$$

The recursion for nominal bond prices with $n \geq 3$ is similar. It is complicated by the fact that we need to integrate over shocks to the inflation target:

$$B_n^s(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \mathbb{E}_t \left[\exp(\log(\beta) - \gamma g + \gamma \hat{s}_t + \gamma \phi x_t - \gamma \hat{s}_{t+1} - \gamma x_{t+1} - \hat{\pi}_{t+1} - n u_{t+1}^* + b_{n-1}^s(\tilde{Z}_{t+1}, \hat{s}_{t+1}, B^s x_t)) \right]. \quad (\text{C.64})$$

To reduce the number of dimensions along which we need to integrate numerically, we split u_{t+1}^* into a component that is spanned by ϵ_{t+1} plus an orthogonal shock. This is useful because we can then use analytic expressions to integrate over the orthogonal component. We use the standard expression for conditional distributions of multivariate normal random variables. The distribution of u_{t+1}^* conditional on ϵ_{t+1} is normal with:

$$u_{t+1}^* | \epsilon_{t+1} \sim N \left(\underbrace{(A Q \Sigma_u e_3')'}_{vec_*} \epsilon_{t+1}, \underbrace{(\sigma^*)^2 - (A Q \Sigma_u e_3')' (A Q \Sigma_u e_3')}_{(\sigma^\perp)^2} \right). \quad (\text{C.65})$$

We then write u_{*t} as the sum of two independent shocks:

$$u_{*t+1} = vec_* \epsilon_{t+1} + \epsilon_{t+1}^\perp, \quad (\text{C.66})$$

where ϵ_{t+1}^\perp is defined as

$$\epsilon_{t+1}^\perp := u_{*t+1} - vec_* \epsilon_{t+1} \quad (\text{C.67})$$

We integrate analytically over ϵ_{t+1}^\perp and substitute in (C.34):

$$\begin{aligned} B_n^s(\tilde{Z}_t, \hat{s}_t, x_{t-1}) &= \mathbb{E}_t \left[\exp(\log(\beta) - \gamma g + \gamma \hat{s}_t + \gamma \phi x_t - \gamma \hat{s}_{t+1} - \gamma x_{t+1} - \hat{\pi}_{t+1} - n v^* \epsilon_{t+1} + \frac{n^2}{2} (\sigma^\perp)^2 + b_{n-1}^s(\tilde{Z}_{t+1}, \hat{s}_{t+1}, B^s x_t)) \right], \\ &= \mathbb{E}_t \left[\exp \left(-\bar{r} - e_3 A^{-1} \tilde{Z}_t - \frac{\gamma}{2} (1 - \theta_0) (1 - 2\hat{s}_t) - (\gamma (1 + \lambda(\hat{s}_t)) \sigma_c + \underbrace{e_2 A^{-1} e_1'}_{v \pi_1} + n v^* e_1') \epsilon_{1,t+1} \right. \right. \\ &\quad \left. \left. - \left(\underbrace{e_2 A^{-1} e_2'}_{v \pi_2} + n v^* e_2' \right) \epsilon_{2,t+1} + \frac{n^2}{2} (\sigma^\perp)^2 + b_{n-1}^s(\tilde{Z}_{t+1}, \hat{s}_{t+1}, B^s x_t) \right) \right]. \quad (\text{C.68}) \end{aligned}$$

We define the vectors vpi_1 and vpi_2 as given above to avoid computing them repeatedly in our numerical algorithm.

C.2.4 Assessing the approximation error for the one-period nominal rate

Throughout the paper, we use the approximation

$$i_t = r_t + E_t \pi_{t+1}. \quad (\text{C.69})$$

This approximation is useful, because it leads to log-linear macroeconomic dynamics in the output gap, inflation, and the nominal interest rate. In order to understand the significance of the error in this approximation, we now derive the exact one-period nominal bond yield while taking the dynamics for real consumption and the real interest rate as given. The exact one-period nominal bond price equals:

$$\begin{aligned} P_{1,t}^{\$,exact} &= \exp(\log(\beta) - \gamma g + \gamma \hat{s}_t + \gamma \phi x_t) \\ &\quad \times \exp(E_t(-\gamma \hat{s}_{t+1} - \gamma x_{t+1} - \hat{\pi}_{t+1} - \pi_{*t+1})) \\ &\quad \times E_t[\exp((-\gamma(\lambda(\hat{s}_t) + 1) Q_M - [e_2 Q + e_3]) u_{t+1})]. \end{aligned} \quad (\text{C.70})$$

We now substitute in (C.34), and take logs to get:

$$\begin{aligned} \log(P_{1,t}^{\$,exact}) &= -r_t - E_t \pi_{t+1} \\ &\quad + \frac{1}{2} (e_2 Q + e_3) \Sigma_u (e_2 Q + e_3)' + \gamma (\lambda(\hat{s}_t) + 1) Q_M \Sigma_u (e_2 Q + e_3)'. \end{aligned} \quad (\text{C.71})$$

We decompose the one-period nominal bond yield into the approximate nominal yield, i_t , and approximation error:

$$\begin{aligned} y_{1,t}^{\$,exact} &= \underbrace{r_t + E_t \pi_{t+1}}_{i_t} \\ &\quad - \underbrace{\frac{1}{2} (e_2 Q + e_3) \Sigma_u (e_2 Q + e_3)' - \gamma (\lambda(\hat{s}_t) + 1) Q_M \Sigma_u (e_2 Q + e_3)'}_{\text{Approximation Error}} \end{aligned} \quad (\text{C.72})$$

At our estimates, we find that the simulated approximation error has a standard deviation of 2bps.

C.2.5 Computing returns

The log return on the consumption claim equals:

$$r_{t+1}^c = \log\left(\frac{P_{t+1}^c + C_{t+1}}{P_t^c}\right), \quad (\text{C.73})$$

$$= \Delta c_{t+1} + \log\left(\frac{1 + \frac{P_{t+1}^c}{C_{t+1}}}{\frac{P_t^c}{C_t}}\right). \quad (\text{C.74})$$

Real and nominal log bond yields equal:

$$y_{n,t} = -\frac{1}{n}b_{n,t}, \quad (\text{C.75})$$

$$y_{n,t}^{\$} = -\frac{1}{n}b_{n,t}^{\$} + \pi_{*t}. \quad (\text{C.76})$$

Real log bond returns equal:

$$r_{n,t+1} = b_{n-1,t+1} - b_{n,t}. \quad (\text{C.77})$$

Nominal log bond returns equal:

$$r_{n,t+1}^{\$} = b_{n-1,t+1}^{\$} - b_{n,t}^{\$} - (n-1)\pi_{*,t+1} + n\pi_{*t}. \quad (\text{C.78})$$

Real and nominal bond log excess returns then equal:

$$xr_{n,t+1} = r_{n,t+1} - r_t, \quad (\text{C.79})$$

$$xr_{n,t+1}^{\$} = r_{n,t+1}^{\$} - i_t. \quad (\text{C.80})$$

C.2.6 Levered stock prices and returns

We note that the price of the levered equity claim is δP_t^c , so the price-dividend ratio equals:

$$\frac{P_t^\delta}{D_t^\delta} = \delta \frac{C_t}{D_t^\delta} \frac{P_t^c}{C_t}. \quad (\text{C.81})$$

Using the expression

$$D_{t+1}^\delta = P_{t+1}^c + C_{t+1} - (1-\delta)P_t^c \exp(r_t) - \delta P_t^c, \quad (\text{C.82})$$

and

$$P_t^\delta = \delta P_t^c \quad (\text{C.83})$$

gives the gross return on levered stocks:

$$(1 + R_{t+1}^\delta) = \frac{D_{t+1}^\delta + P_{t+1}^\delta}{P_t^\delta}, \quad (\text{C.84})$$

$$= \frac{1}{\delta} \frac{P_{t+1}^c + C_{t+1} - (1 - \delta)P_t^c \exp(r_t)}{P_t^c}, \quad (\text{C.85})$$

$$= \frac{1}{\delta} (1 + R_{t+1}^c) - \frac{1 - \delta}{\delta} \exp(r_t). \quad (\text{C.86})$$

Log stock excess returns then equal:

$$xr_{t+1}^\delta = r_{t+1}^\delta - r_t. \quad (\text{C.87})$$

To mimic firms' dividend smoothing in the data, we report simulated moments for the price of equities dividend by dividends smoothed over the past 64 quarters:

$$P_t^\delta / \left(\frac{1}{64} (D_t^\delta + D_{t-1}^\delta + \dots + D_{t-63}^\delta) \right). \quad (\text{C.88})$$

C.3 Details: Risk-premium decomposition

For the risk-premium decomposition in Table 7 in the main paper, we use the following steps. We first compute risk-neutral valuations and returns. We then decompose risk-neutral returns further into news about the real interest rate and news about cash flows using the Campbell and Ammer (1993) loglinear expressions. We then define returns due to risk premia as log asset excess returns minus cash-flow news. We use the superscript rn for risk-neutral, superscript cf for cash flow, and rp for risk premium.

Risk-neutral valuations are expected cash flows discounted with the risk-neutral discount factor, that is consistent with equilibrium dynamics for the real interest rate:

$$M_{t+1}^{rn} = \exp(-r_t) \quad (\text{C.89})$$

$$= \exp(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r}). \quad (\text{C.90})$$

C.3.1 Risk-neutral zero-coupon bond prices

We use analogous recursions to solve for risk-neutral bond prices. One-period risk-neutral bond prices are given exactly as before:

$$P_{1,t}^{\$,rn} = \exp(-\hat{i}_t - \pi_t^* - \bar{r}), \quad (\text{C.91})$$

$$P_{1,t}^{rn} = \exp(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r}). \quad (\text{C.92})$$

For $n > 1$, we guess and verify that the prices of real and nominal risk-neutral zero-coupon bonds with maturity n can be written in the following form

$$P_{n,t}^{rn} = B_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}), \quad (\text{C.93})$$

$$P_{n,t}^{\$,rn} = \exp(-n\pi_t^*) B_n^{\$,rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}). \quad (\text{C.94})$$

for some functions $B_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1})$ and $B_n^{\$,rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1})$.

We derive the two-period risk-neutral nominal bond price analytically:

$$P_{2,t}^{\$,rn} = \exp(-r_t) \mathbb{E}_t \left[P_{1,t+1}^{\$,rn} \exp(-\pi_{t+1}^* - \hat{\pi}_{t+1}) \right] \quad (\text{C.95})$$

$$= \exp(-r_t) \mathbb{E}_t \left[\exp(-\hat{i}_{t+1} - 2\pi_{t+1}^* - \hat{\pi}_{t+1} - \bar{r}) \right]. \quad (\text{C.96})$$

We can hence verify that the two-period risk-neutral nominal bond price takes the form (C.52) with:

$$\begin{aligned} B_2^{\$,rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) &= \exp\left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r}\right) \exp\left(\mathbb{E}_t\left(-\hat{i}_{t+1} - \hat{\pi}_{t+1} - \bar{r}\right)\right) \\ &\quad \times \mathbb{E}_t \left[\exp\left(\left(\underbrace{-[(e_2 + e_3)Q + 2e_3]}_{v_{\$}}\right) u_{t+1}\right) \right]. \end{aligned} \quad (\text{C.97})$$

Here, the vector $v_{\$}$ is identical to the case with risk aversion. Taking logs, we get:

$$b_2^{\$,rn} = -e_3 [I + P] A^{-1} \tilde{Z}_t + \frac{1}{2} v_{\$} \Sigma_u v_{\$}' - 2\bar{r} \quad (\text{C.98})$$

Comparing expressions (C.98) and (C.59) shows that they agree when $\gamma = 0$. We similarly solve for 2-period real bond prices in closed form:

$$\begin{aligned} P_{2,t}^{rn} &= \exp\left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r}\right) \times \exp\left(\mathbb{E}_t\left(-\hat{i}_{t+1} + \mathbb{E}_{t+1} \hat{\pi}_{t+2} - \bar{r}\right)\right) \\ &\quad \times \mathbb{E}_t \left[\exp\left(\underbrace{-(e_3 - e_2 P)Q}_{v_r} u_{t+1}\right) \right]. \end{aligned} \quad (\text{C.99})$$

The vector v_r is again identical to the case with risk aversion. Taking logs gives:

$$b_2^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = -(e_3 - e_2 P) [I + P] A^{-1} \tilde{Z}_t + \frac{1}{2} v_r \Sigma_u v_r' - 2\bar{r}. \quad (\text{C.100})$$

We note that the risk-neutral bond prices (C.100) and bond prices with risk aversion (C.61) are identical when the utility curvature parameter γ equals zero.

For $n \geq 3$ the n -period risk neutral real bond price B_n^{rn} satisfies the recursion:

$$B_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \mathbb{E}_t \left[\exp\left(-\bar{r} - (e_3 - e_2 P) A^{-1} \tilde{Z}_t + b_{n-1}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t)\right) \right] \quad (\text{C.101})$$

We obtain a similar recursion for risk-neutral nominal bond prices:

$$B_n^{\$,rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \mathbb{E}_t \left[\exp \left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r} - \hat{\pi}_{t+1} - nu_{t+1}^* + b_{n-1}^{\$}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, B^{\$}x_t) \right) \right].$$

We again use the decomposition $u_{*t+1} = \text{vec}_* \epsilon_{t+1} + \epsilon_{t+1}^\perp$ from Section C.2.3 to reduce the dimensionality of the numerical integration:

$$B_n^{\$,rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \mathbb{E}_t \left[\exp \left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r} - \hat{\pi}_{t+1} - n \cdot \text{vec}_* \epsilon_{t+1} + \frac{n^2}{2} (\sigma^\perp)^2 + b_{n-1}^{\$}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, B^{\$}x_t) \right) \right], \quad (\text{C.102})$$

$$= \mathbb{E}_t \left[\exp \left(-\bar{r} - e_3 A^{-1} \tilde{Z}_t - \underbrace{(e_2 A^{-1} e'_1 + n \cdot \text{vec}_* e'_1)}_{vpi_1} \epsilon_{1,t+1} - \left(\underbrace{e_2 A^{-1} e'_2 + n \cdot \text{vec}_* e'_2}_{vpi_2} \right) \epsilon_{2,t+1} + \frac{n^2}{2} (\sigma^\perp)^2 + b_{n-1}^{\$}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, B^{\$}x_t) \right) \right]. \quad (\text{C.103})$$

C.3.2 Risk-neutral zero-coupon consumption claims

Next, we derive recursive solutions for the risk-neutral prices of zero-coupon consumption claims. Let $P_{nt}^{c,rn}/C_t$ denote the risk-neutral price-dividend ratio of a zero-coupon claim on consumption at time $t+n$. The risk-neutral price-consumption ratio of a claim to the entire stream of future consumption equals:

$$\frac{P_t^{c,rn}}{C_t} = \sum_{n=1}^{\infty} \frac{P_{nt}^{c,rn}}{C_t}. \quad (\text{C.104})$$

For $n \geq 1$, we guess and verify there exists a function $F_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1})$, such that

$$\frac{P_{nt}^{c,rn}}{C_t} = F_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}). \quad (\text{C.105})$$

We start by deriving the analytic expression for F_1^{rn} . The one-period risk-neutral zero-coupon price-consumption ratio solves

$$\frac{P_{1,t}^{c,rn}}{C_t} = \exp \left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r} \right) \mathbb{E}_t \left[\frac{C_{t+1}}{C_t} \right] \quad (\text{C.106})$$

Using (C.23) to substitute for consumption growth, we can derive the following analytic expression for f_1^{rn} :

$$\begin{aligned} f_1^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) &= -\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r} + g - \phi x_t + E_t x_{t+1} + \frac{1}{2} \sigma_c^2, \\ &= -(e_3 - e_2 P) A^{-1} \tilde{Z}_t - \bar{r} + g + e_1 [P - \phi I] A^{-1} \tilde{Z}_t + \frac{1}{2} \sigma_c^2. \end{aligned} \quad (\text{C.107})$$

Next, we solve for f_n , $n \geq 2$ iteratively:

$$\frac{P_{nt}^{c, rn}}{C_t} = \exp\left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r}\right) \mathbb{E}_t \left[\frac{C_{t+1}}{C_t} F_{n-1}^{rn} \left(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t \right) \right] \quad (\text{C.108})$$

This gives the following expression for f_n^{rn} :

$$f_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \log \left[\mathbb{E}_t \left[\exp \left(-\hat{i}_t + \mathbb{E}_t \hat{\pi}_{t+1} - \bar{r} + g - \phi x_t + \mathbb{E}_t x_{t+1} + \sigma_c \epsilon_{1,t+1} + f_{n-1}^{rn}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t) \right) \right] \right]. \quad (\text{C.109})$$

Finally, we re-write $f_{n,t}^{rn}$ as an expectation involving $f_{n-1,t+1}^{rn}$, the state variables \tilde{Z}_t , and period $t+1$ shocks:

$$f_n^{rn}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \log \left[\mathbb{E}_t \left[\exp \left(g + e_1 [P - \phi I] A^{-1} \tilde{Z}_t - \bar{r} - (e_3 - e_2 P) A^{-1} \tilde{Z}_t + \sigma_c \epsilon_{1,t+1} + f_{n-1}^{rn}(\tilde{Z}_{t+1}, \hat{s}_{t+1}, x_t) \right) \right] \right]. \quad (\text{C.110})$$

C.3.3 Risk-neutral returns

We plug risk-neutral price-consumption ratios and bond prices into equations (C.74) through (C.80). This gives risk-neutral returns on the consumption claim, risk-neutral log excess bond returns, and risk-neutral bond yields. We then substitute risk-neutral returns on the consumption claim into (C.86)-(C.87) to obtain risk-neutral log excess stock returns.

C.3.4 Cash-flow news, real-rate news, and risk-premium excess returns

We decompose risk-neutral returns further into cash-flow news and real-rate news. We use the approximate log-linear decomposition of Campbell and Shiller (1988):

$$x r_{t+1}^{\delta, rn} = r_{t+1}^{\delta, rn} - r_t, \quad (\text{C.111})$$

$$= r_{t+1}^{\delta, rn} - E_t r_{t+1}^{\delta, rn} \quad (\text{C.112})$$

$$= \underbrace{(E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j}}_{\text{Cash Flow News}} - \underbrace{(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+j}}_{\text{Real Rate News}}. \quad (\text{C.113})$$

The first equality follows, because the risk neutral SDF ensures that expected risk neutral returns are equal to the real risk-free rate r_t . Here, ρ is the log-linearization constant corresponding to the risk-neutral price-dividend ratio

$$\rho = \frac{1}{1 + \exp(\text{mean}(pd^{\delta, rn}))}, \quad (\text{C.114})$$

and $mean(pd^{rn})$ denotes the average risk-neutral log price-dividend ratio, which in practice we obtain from 2 model simulations of length 10000.

We use the following analytic expression for the equity excess returns due to real-rate news:

$$xr_{t+1}^{\delta,rr} = -(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+j} \quad (C.115)$$

$$= -\rho (e_3 - e_2 P) (I - \rho P)^{-1} Qu_{t+1}. \quad (C.116)$$

We obtain cash-flow news as risk-neutral excess returns minus returns due to real-rate news:

$$xr_{t+1}^{\delta,cf} = xr_{t+1}^{\delta,rn} + \rho (e_3 - e_2 P) (I - \rho P)^{-1} Qu_{t+1} \quad (C.117)$$

The risk-premium component of equity excess returns equals log excess stock returns minus risk-neutral excess returns:

$$xr_{t+1}^{\delta,rp} = xr_{t+1}^{\delta} - xr_{t+1}^{\delta,rn}. \quad (C.118)$$

We similarly decompose bond returns into cash-flow news, real-rate news, and risk-premium excess returns, using the log-linear exact expression from Campbell and Ammer (1993):

$$xr_{n,t+1}^{\$,rn} = r_{t+1}^{\$,rn} - i_t, \quad (C.119)$$

$$= r_{n,t+1}^{\$,rn} - E_t r_{n,t+1}^{\$,rn}, \quad (C.120)$$

$$= (E_{t+1} - E_t) \left\{ -\sum_{i=1}^{n-1} \pi_{t+1+i} - \sum_{i=1}^{n-1} r_{t+1+i} \right\}. \quad (C.121)$$

The real-rate news component of nominal bond log excess returns is simply the risk-neutral log excess return on a real n -period bond

$$xr_{n,t+1}^{\$,rr} = (E_{t+1} - E_t) \sum_{i=1}^{n-1} r_{t+1+i}, \quad (C.122)$$

$$= xr_{t+1}^{rn}. \quad (C.123)$$

With this, we compute the cash-flow news component of nominal bond returns:

$$xr_{n,t+1}^{\$,cf} = xr_{t+1}^{\$,rn} - xr_{t+1}^{rn}. \quad (C.124)$$

The risk-premium component of log excess nominal bond returns is defined as the log excess return minus the risk-neutral log excess return:

$$xr_{t+1}^{\$,dr} = xr_{t+1}^{\$} - xr_{t+1}^{\$,rn}. \quad (C.125)$$

D Details: Numerical algorithm

Since our preferences nest CC, a minimum requirement for any numerical solution method is that it must be accurate for the original CC model and calibration. We evaluate asset prices by iterating on a grid for the state vector following Wachter (2005). Other numerical methodologies are faster, but their cost is that they cannot replicate the economic properties of Wachter (2005)'s numerical solution for CC. In unreported results, we verified that analytic linear approximations to the sensitivity function λ (e.g. Lopez, López-Salido, and Vazquez-Grande 2014), numerical higher-order perturbation methods using Dynare (Rudebusch and Swanson 2008), and global projection methods give solutions for CC that are economically very different from Wachter (2005)'s numerical solution.

Other approaches in the literature are also not appropriate for our problem. While Chen (2017) solves a model with habit and production using global projection and perturbation methods, his model features a linear sensitivity function and heteroskedastic consumption. By contrast, we have homoskedastic consumption and a highly non-linear sensitivity function. Similarly, affine term structure models, such as Dai and Singleton (2000), generate affine relations between risk premia and state variables by assuming analytically convenient functional forms for the pricing kernel. In contrast to models that assume more convenient pricing kernels, our preferences are consistent with the standard log-linear New Keynesian consumption Euler equation and generate conditionally homoskedastic macroeconomic dynamics.

While iterating on a grid is significantly slower than perturbation or global projection methods, it is not prohibitively so. Our MATLAB algorithm for solving the asset pricing recursions (described in Section D.1) takes 80 seconds to run on a Lenovo X270 laptop with an i7-7600 CPU. Simulating the model (described in Section D.2) takes 11 seconds. The risk-neutral asset pricing recursions and simulating the risk-neutral stock returns take an additional 80 seconds and 11 seconds. MATLAB is not a particularly efficient programming language, so it is plausible that further speed ups are possible by using a lower-level programming language, such as FORTRAN or C.

D.1 Implementing the asset pricing recursions

We implement the recursions in Sections C.2.2 and C.2.3 numerically through value function iteration on a grid. We solve for the functions f_n , b_n , and b_n^s using value function iteration along a five-dimensional state vector. We use a five-dimensional grid, with the first three dimensions corresponding to \tilde{Z}_t , the fourth dimension corresponding to \hat{s}_t , and the fifth dimension corresponding to x_{t-1} .

D.1.1 Grid

In this section, we use \tilde{Z}, \hat{s}, x to denote the corresponding time- t variables. We use superscripts $-$ to denote variables in the previous period and $+$ to denote variables in the next period. We solve numerically for $f_n, b_n,$ and $b_n^{\$}$ as functions of the vector of state variables $[\tilde{Z}, \hat{s}, x^-]$.

Our grid is densest along the \hat{s} dimension to capture important non-linearities of asset prices with respect to the surplus consumption ratio. Following Wachter (2005), we choose a grid for the surplus consumption ratio that consists of an upper segment and a lower segment and covers a wide range of values for s_t . Let $S_{grid,1}$ denote a vector of 20 equally spaced points between 0 and S_{max} with S_{max} included and $s_{grid,2}$ a vector of 30 equally spaced points between $\min(\log(S_{grid,1}))$, and -50 . The grid for $\hat{s}_t = s_t - \bar{s}$ then consists of the concatenation of $s_{grid,2} - \bar{s}$ and $\log(S_{grid,1}) - \bar{s}$.

We find that bond and stock prices are close to loglinear in \tilde{Z} and \hat{x}^- , so coarser grids are sufficient along those dimensions of the state vector. In fact, the analytic expressions for $f_1, b_2,$ and $b_2^{\$}$ show that one-period zero-coupon consumption claims and two-period bond prices are exactly log-linear in \tilde{Z} and x^- . Numerical results indicate that this property translates to longer-period claims and $f_n, b_n,$ and $b_n^{\$}$ are still approximately linear in \tilde{Z} and x^- for general n . To speed up the value function iteration, we therefore use two grid points for each dimension of \tilde{Z} and for x^- .

For \tilde{Z} , we use an equal-spaced three-dimensional grid. Let N denote the number of grid points along each dimension and m the width of the grid as a multiple of the unconditional standard deviation of \tilde{Z} . For each dimension of \tilde{Z} , we choose a grid of N equal-spaced points with the lowest point equal to $-m \times std(\tilde{Z})$ and the highest point equal to $m \times std(\tilde{Z})$. Here, the unconditional variance-covariance matrix of \tilde{Z} is determined implicitly by the equation:

$$std(\tilde{Z}) = \sqrt{\tilde{P}Var(\tilde{Z})\tilde{P}' + diag(1, 1, 0)}. \quad (D.1)$$

For our baseline grid, we set $N = 2$ and $m = 2$.

For x^- , we consider an equal-spaced grid with $sizexm$ points ranging from $\min(e_1 A \tilde{Z}_t : \tilde{Z} \in grid)$ to $\max(e_1 A \tilde{Z} : \tilde{Z} \in grid)$. This choice of grid ensures that the grid for x^- covers the entire range of output gap values implied by the grid for \tilde{Z} . In our baseline evaluation, we set $sizexm = 2$.

With $N = 2$ grid points along each of the three dimensions of \tilde{Z} , 50 gridpoints for \hat{s} , and $sizexm = 2$ grid points for x^- , the combined grid has a total of $2^3 \cdot 50 \cdot 2 = 800$ points.

D.1.2 Numerical integration

Following Wachter (2005), we use Gauss-Legendre quadrature to evaluate the expectations (C.50), (C.63), and (C.68) numerically. Gauss-Legendre quadrature is orders of magnitude faster than computing expectations by simulation. As in Wachter (2005), we evaluate infinite integrals over the density of standardized consumption shocks ($\epsilon_{1,t}$) using 40 integration node points and an integration domain ranging from -8 standard deviations to $+8$ standard deviations. To conserve speed and memory, we integrate over shocks orthogonal to surplus consumption ($\epsilon_{2,t}$) using a somewhat smaller number of integration node points, 15, but again an integration domain of ± 8 standard deviations. To evaluate bond and stock prices at points that are not on the grid, we use loglinear multi-linear interpolation and extrapolation.

For completeness, we recap the key features of Gauss-Legendre integration. Let xGL_i , $i = 1, \dots, N_{GL}$ and $wGL_i = 1, \dots, N_{GL}$ denote the Gauss-Legendre nodes and weights of N_{GL} th order. Gauss-Legendre quadrature then approximates a definite integral of any smooth function f on the interval $[-1, 1]$ by $\int_{-1}^1 f(x) dx \approx \sum_{i=1}^{N_{GL}} wGL_i f(xGL_i)$. By change of variable, it is immediate that we can approximate the integral of a smooth function f on an interval $[-\bar{a}, \bar{a}]$ by

$$\int_{-\bar{a}}^{\bar{a}} f(x) dx \approx \sum_{i=1}^{N_{GL}} \underbrace{\bar{a} \times wGL_i}_{wGL_i^{\bar{a}}} f\left(\underbrace{\bar{a} \times xGL_i}_{xGL_i^{\bar{a}}}\right). \quad (\text{D.2})$$

Here, we use $xGL_i^{\bar{a}}$ and $wGL_i^{\bar{a}}$ to denote Gauss-Legendre node points and weights scaled to the interval $[-\bar{a}, \bar{a}]$.

We implement Gauss-Legendre quadrature to take expectations over ϵ_{t+1} as follows. Let N_1 denote the number of Gauss-Legendre nodes and \bar{a}_1 denote the integration domain for the shock $\epsilon_{1,t}$, that is perfectly correlated with output innovations. We set $xGL_{1,i} = xGL_i^{\bar{a}_1}$ and $wGL_{1,i} = wGL_i^{\bar{a}_1}$ for $i = 1, \dots, N_1$, where the weights and nodes are as defined in equation (D.2). Moreover, we set

$$pGL_{1,i} = \frac{1}{\sqrt{2\pi}} \exp(-xGL_{1,i}^2) wGL_{1,i} / \sum_{i=1}^{N_1} \left(\frac{1}{\sqrt{2\pi}} \exp(-xGL_{1,i}^2) wGL_{1,i} \right), \quad (\text{D.3})$$

and use the scaled weights $pGL_{1,i}$ for numerical integration. The scaling of (D.3) ensures that the numerical expectation of a constant is evaluated to be the same constant (or intuitively that discretized probabilities sum to one).

We then evaluate numerically the expectation of any smooth function f of $\epsilon_{1,t}$ via:

$$E[f(\epsilon_{1,t})] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\epsilon_1^2) f(\epsilon_1) d\epsilon_1, \quad (\text{D.4})$$

$$\approx \int_{-\bar{a}_1}^{\bar{a}_1} \frac{1}{\sqrt{2\pi}} \exp(-\epsilon_1^2) f(\epsilon_1) d\epsilon_1, \quad (\text{D.5})$$

$$\approx \sum_{i=1}^{N_1} pGL_{1,i} f(xGL_{1,i}). \quad (\text{D.6})$$

Accuracy increases with \bar{a}_1 and N_1 . We follow Wachter (2006) in setting $N_1 = 40$ and $\bar{a}_1 = 8$.

To take expectations over $\epsilon_{2,t}$, we similarly use Gauss-Legendre quadrature with integration domain $\bar{a}_2 = 8$ and number of nodes $N_2 = 15$. We set $xGL_{2,i} = xGL_i^{\bar{a}_2}$ and $wGL_{2,i} = wGL_i^{\bar{a}_2}$ for $i = 1, \dots, N_2$ and define the scaled weights:

$$pGL_{2,i} = \frac{1}{\sqrt{2\pi}} \exp(-xGL_{2,i}^2) wGL_{2,i} / \sum_{i=1}^{N_2} \left(\frac{1}{\sqrt{2\pi}} \exp(-xGL_{2,i}^2) wGL_{2,i} \right), \quad (\text{D.7})$$

Since $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are independent, we can evaluate the expectation of any smooth function $f(\epsilon_{1,t}, \epsilon_{2,t})$ as

$$Ef(\epsilon_{1,t}, \epsilon_{2,t}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\epsilon_2^2) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\epsilon_1^2) f(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2, \quad (\text{D.8})$$

$$\approx \sum_{i=1}^{N_2} pGL_{2,i} \left[\sum_{j=1}^{N_1} pGL_{1,j} f(xGL_{1,i}, xGL_{2,j}) \right]. \quad (\text{D.9})$$

D.1.3 Recursive step

Let a superscript num denote the numerical counterparts to the analytic functions f_n , b_n , $b_n^{\$}$. We start by initializing $f_1^{num}(\tilde{Z}, \hat{s}, x^-)$, $b_2^{num}(\tilde{Z}, \hat{s}, x^-)$, and $b_2^{\$,num}(\tilde{Z}, \hat{s}, x^-)$ at each grid point according to the analytic expressions (C.47), (C.59) and (C.61).

Next, we apply the recursive expressions (C.50), (C.63), and (C.68) along the grid. Having computed f_{n-1}^{num} along the entire grid, we evaluate $f_n^{num}(\tilde{Z}, \hat{s}, x^-)$ at a grid

point $(\tilde{Z}, \hat{s}, x^-)$ as follows. We compute the expectation (C.50) numerically as:

$$\begin{aligned}
f_n^{num}(\tilde{Z}, \hat{s}, x^-) = & \log \left[\sum_{j=1}^{N2} pGL_{2,j} \left[\sum_{i=1}^{N1} pGL_{1,i} \cdot \exp \left(g + e_1[P - \phi I]A^{-1}\tilde{Z} \right. \right. \right. \\
& - \bar{r} - (e_3 - e_2P)A^{-1}\tilde{Z} - \frac{\gamma}{2}(1 - \theta_0)(1 - 2\hat{s}) \\
& \left. \left. \left. - (\gamma(1 + \lambda(\hat{s})) - 1)\sigma_c \times xGL_{1,i} \right. \right. \right. \\
& \left. \left. \left. + f_{n-1}^{num} \left(\tilde{P}\tilde{Z} + \begin{bmatrix} xGL_{1,i} \\ xGL_{2,j} \\ 0 \end{bmatrix}, \theta_0\hat{s} + \theta_1x + \theta_2x^- + \lambda(\hat{s})xGL_{1,i}, x \right) \right] \right] \right], \tag{D.10}
\end{aligned}$$

where we evaluate x as a function of the state vector as

$$x = e_1A^{-1}\tilde{Z}. \tag{D.11}$$

To compute the right-hand-side of (D.10), we need to evaluate f_{n-1}^{num} at points that are not on our grid. We interpolate f_{n-1}^{num} linearly (and hence F_{n-1}^{num} log-linearly). When the argument is outside the range of the grid, we extrapolate f_{n-1}^{num} linearly. It is clear from (C.47) that linear inter- and extrapolation gives a good approximation of f_1 . In fact, we can see that f_1 is exactly linear in \tilde{Z} , independent of x^- , and that it depends on $\lambda(\hat{s}) = \lambda_0\sqrt{1 - 2\hat{s}}$. We accommodate the fact that f_1 is not linear in \hat{s} by choosing a much denser grid along the \hat{s} dimension. We do not have analytic expressions for $f_n, n > 1$ (after all, that's why we need a numerical solution), but numerical solutions indicate that linear inter- and extrapolation gives good approximations for f_n with the chosen grid.

In terms of coding (D.10), we face a trade-off between speed and readability of the code. We pre-allocate matrices outside loops and we use code linear interpolation by hand (rather than using a pre-written interpolation routine) to conserve speed and memory. We also inline the linear interpolation steps (i.e. write them directly into the main function rather than calling a separate interpolation function). This speeds up the code substantially, while somewhat reducing its readability.

There are different methods to interpolate multidimensional functions. Specifically, we use multi-linear interpolation, corresponding to interpolating along each dimension one at a time. In order to enhance computational speed we do not rely on a pre-programmed interpolation routine, instead coding our own minimal interpolation routine. It is well-known that the result of multi-linear (or in the two-dimensional case bi-linear) interpolation does not depend on in which order one interpolates the different arguments. We find it convenient to interpolate $f_{n-1}^{num}(\tilde{Z}, \hat{s}, x^-)$ first along the x^- dimension, then along \hat{s} , then along \tilde{Z}_1 , and finally along the \tilde{Z}_2 and \tilde{Z}_3 dimensions.

Finally, we evaluate the price-consumption ratio for the aggregate consumption stream by approximating it as the sum of the first 300 zero-coupon consumption

claims:

$$G^{num}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = \sum_{n=1}^{300} \exp\left(f_n^{num}(\tilde{Z}_t, \hat{s}_t, x_{t-1})\right). \quad (\text{D.12})$$

We iterate $b_n^{num}(\tilde{Z}, \hat{s}, x^-)$ and $b_n^{s,num}(\tilde{Z}, \hat{s}, x^-)$ similarly according to:

$$\begin{aligned} b_n^{num}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = & \log \left[\sum_{j=1}^{N2} pGL_{2,j} \left[\sum_{i=1}^{N1} pGL_{1,i} \cdot \exp\left(-\bar{r} - (e_3 - e_2 P) A^{-1} \tilde{Z} - \frac{\gamma}{2}(1 - \theta_0)(1 - 2\hat{s}) \right. \right. \right. \\ & \left. \left. \left. - \gamma(1 + \lambda(\hat{s})) \sigma_c \times xGL_{1,i} \right) \right] \right. \\ & \left. \left. + b_{n-1}^{num} \left(\tilde{P} \tilde{Z} + \begin{bmatrix} xGL_{1,i} \\ xGL_{2,j} \\ 0 \end{bmatrix}, \theta_0 \hat{s} + \theta_1 x + \theta_2 x^- + \lambda(\hat{s}) xGL_{1,i}, x \right) \right] \right], \quad (\text{D.13}) \end{aligned}$$

and

$$\begin{aligned} b_n^{s,num}(\tilde{Z}_t, \hat{s}_t, x_{t-1}) = & \log \left[\sum_{j=1}^{N2} pGL_{2,j} \left[\sum_{i=1}^{N1} pGL_{1,i} \cdot \exp\left(-\bar{r} - e_3 A^{-1} \tilde{Z} - \frac{\gamma}{2}(1 - \theta_0)(1 - 2\hat{s}) \right) \right. \right. \\ & \left. \left. - (\gamma(1 + \lambda(\hat{s})) \sigma_c + vpi_1 + n \cdot \text{vec}_* e'_1) \times xGL_{1,i} \right) \right] \\ & \left. - (vpi_2 + n \cdot \text{vec}_* e'_2) xGL_{2,j} + \frac{n^2}{2} (\sigma^\perp)^2 \right. \\ & \left. \left. + b_{n-1}^{s,num} \left(\tilde{P} \tilde{Z} + \begin{bmatrix} xGL_{1,i} \\ xGL_{2,j} \\ 0 \end{bmatrix}, \theta_0 \hat{s} + \theta_1 x + \theta_2 x^- + \lambda(\hat{s}) xGL_{1,i}, x \right) \right] \right], \quad (\text{D.14}) \end{aligned}$$

We again use multi-linear interpolation and extrapolation to evaluate $b_{n-1}^{s,num}$ and b_{n-1}^{num} at points that are not on the grid.

We similarly implement the recursions (C.101), (C.103), and (C.110) numerically to obtain risk-neutral bond and consumption claim valuations $B_n^{rn,num}$, $B_n^{rn,s,num}$, $G^{rn,num}$.

D.2 Simulating the Model

We simulate a draw of length T . Reported results in Tables 2 through 6 use $T = 10000$ and discard the first 100 simulation periods to ensure that the system has reached the stochastic steady-state. Tables 2 through 6 report model moments averaged across 2 independent simulations.

We use superscript *sim* to denote simulated quantities. We use the MATLAB function `mvrnd` to draw

$\epsilon_1^{sim}, \dots, \epsilon_T^{sim} \stackrel{iid}{\sim} N(0, \text{diag}(1, 1, 0))$. We similarly generate independent draws for the orthogonal shock $\epsilon_1^{\perp, sim}, \dots, \epsilon_T^{\perp, sim} \stackrel{iid}{\sim} N(0, \sigma^\perp)$. We generate draws for u_{*t}^{sim} by plugging ϵ_t^{sim} and $\epsilon_t^{\perp, sim}$ into (C.66). We generate draws for $\tilde{Z}_t^{sim}, t = 1, \dots, T$ by setting $\tilde{Z}_1^{sim} = 0$ and then updating according to (C.36). This gives the simulated output gap, inflation gap, and interest rate gap for $t = 1, 2, \dots, T$ through the relation $\begin{bmatrix} x_t^{sim} \\ \hat{\pi}_t^{sim} \\ \hat{i}_t^{sim} \end{bmatrix} = \hat{Y}_t^{sim} = A^{-1} \tilde{Z}_t^{sim}$. We generate draws for the surplus consumption ratio by setting $\hat{s}_1^{sim} = 0$ and $x_0^{sim} = 0$ and then updating according to (C.28). We generate the simulated inflation target series $\pi_{*t}, t = 1, 2, \dots, T$ by starting from $\pi_{*1}^{sim} = 0$ and updating it according to equation (17) in the main paper. We initialize simulated log consumption at $c_1^{sim} = 0$ and update it using (C.23). We then drop the first 100 simulation periods to allow the system to converge to the stochastic steady-state.

Having generated draws for the five state variables $\tilde{Z}^{sim}, \hat{s}^{sim}$, and x_{t-1}^{sim} , we obtain the simulated consumption-claim price-dividend ratio as $(P^c/C)_t^{sim} = G^{num}(\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim})$, n -period real bond prices as $P_{n,t}^{sim} = B_n^{num}(\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim})$, and $B_{n,t}^{\$,sim} = B_n^{\$,num}(\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim})$. We obtain the corresponding risk-neutral valuation ratios by plugging into the risk-neutral asset pricing solutions: $(P^c/C)_t^{rn,sim} = G^{rn,num}(\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim})$, $P_{n,t}^{rn,sim} = B_n^{rn,num}(\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim})$, and $B_{n,t}^{rn,\$,sim} = B_n^{rn,\$,num}(\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim})$. We obtain nominal bond prices $P_{n,t}^{\$,sim}$ by combining $B_{n,t}^{\$,sim}$ and u_{*t}^{sim} according to (C.52). We similarly obtain risk-neutral nominal bond prices $P_{n,t}^{rn,\$,sim}$ by combining $B_{n,t}^{rn,\$,sim}$ and u_{*t}^{sim} according to (C.52).

To deal with the fact that $\tilde{Z}_t^{sim}, \hat{s}_t^{sim}, x_{t-1}^{sim}$ are not usually on grid points we adopt a similar linear interpolation strategy as in the numerical evaluation of the asset pricing recursions described in Section D.1.3. We interpolate G^{num}, B_n^{num} , and $B_n^{\$,num}$ log-linearly. We simplify the interpolation strategy slightly compared to Section D.1.3. We use the MATLAB function `griddedInterpolant`, sacrificing some computational speed for simpler code. Even though rare events (and especially extremely negative realizations for \hat{s}) matter for the value function iteration in Section D.1.3, low-probability events have very little impact on the properties of simulated asset prices taking as given G^{num}, B_n^{num} , and $B_n^{\$,num}$. We therefore simplify the log-linear interpolation by truncating $\tilde{Z}_t^{sim}, \hat{s}_t^{sim}$, and x_{t-1}^{sim} at the maximum and minimum values covered by the grid.

Having generated $(\frac{P^c}{C})_t^{sim}, t = 1, \dots, T$, we compute log returns on the consumption claim $r_{t+1}^{c,sim}$ according to (C.74). We obtain simulated price-dividend ratios for levered stocks by plugging into (C.81). Finally, we obtain log bond yields and stock and bond excess returns as described in Section C.2.5. Risk-neutral bond and stock

returns are computed by substituting $(\frac{P^c}{C})^{rn,sim}$, $P_{n,t}^{rn,\$,sim}$, and $P_{n,t}^{rn,sim}$ into the same relations.

We obtain cash-flow news, real-rate news, and risk-premium excess returns for nominal bonds and levered stocks by substituting simulated returns and real rates into the expression in Section C.3.4. In our simulations, it is possible but very rare (less than 1 in 20000 simulation periods) that the levered price-dividend ratio turns negative. In that event, we discard the simulation run and simulate the model again.

D.3 Parameter units

This subsection details the relation between parameter values in empirical (reported in the paper) and natural units (used for solving the code). We solve the model in natural units. However, it is most natural to report empirical moments and summary statistics in empirical units for interpretability.

For comparability with empirical moments, Table 1 reports model parameters in units that correspond to the output gap in annualized percent, and inflation and interest rates in annualized percent. For comparability with CC, we report the discount rate and the persistence of surplus consumption in annualized units. Concretely, Table 1 reports the following scaled parameters:

$$400 \times g, \tag{D.15}$$

$$400 \times \bar{r}, \tag{D.16}$$

$$\theta_0^4, \tag{D.17}$$

$$\beta^4, \tag{D.18}$$

$$\frac{1}{4} \times \psi, \tag{D.19}$$

$$400 \times \sigma_\pi, \tag{D.20}$$

$$400 \times \sigma_i, \tag{D.21}$$

$$400 \times \sigma_*, \tag{D.22}$$

$$4 \times p_{\pi x}, \tag{D.23}$$

$$4 \times p_{ix} \tag{D.24}$$

All other parameters reported in Table 1 do not need to be scaled.

E Details: Econometric methodology

E.1 Impulse response functions

This section describes how we estimate the macroeconomic impulse responses, that we use for the SMM estimation and are shown in Figures 3 through 5. We follow the procedure described below for both actual and simulated data, with the simulated data length matching the length of the empirical sample. Model impulse responses in Figures 3 through 5 are averaged over 100 simulations. In this section, we use subscripts $_{IRF}$ if variable names would otherwise be similar to different variables elsewhere in the paper.

We estimate a VAR(1) of the form

$$Y_{IRF,t} = \Pi Y_{IRF,t-1} + \varepsilon_t, \quad (\text{E.1})$$

where we define the vector for the VAR(1) as:

$$Y_{IRF,t} = [x_{t-1}, \pi_t, i_t]. \quad (\text{E.2})$$

The shocks ε_t are not orthogonal and we denote their estimated variance-covariance matrix by Σ_ε .

Next, we rotate the innovations to be orthogonal. This means that we need to re-write the VAR(1) in the form:

$$R^{-1}Y_{IRF,t} = \Pi_R Y_{IRF,t-1} + \eta_t, \quad (\text{E.3})$$

where η_t is a vector of uncorrelated shocks, R is an invertible matrix, and $\Pi_R = R^{-1}\Pi$. We write the variance-covariance matrix of η_t as:

$$\Sigma_\eta = E\eta_t'\eta_t, \quad (\text{E.4})$$

$$= \begin{bmatrix} \sigma(\eta_1)^2 & 0 & 0 \\ 0 & \sigma(\eta_2)^2 & 0 \\ 0 & 0 & \sigma(\eta_3)^2 \end{bmatrix}. \quad (\text{E.5})$$

There are many ways of writing the VAR(1) in the form (E.3). Following Sims (1986), we pick a unique representation by requiring R^{-1} to be lower-diagonal with ones along the diagonal. Having estimated Π and Σ_ε , we obtain R , Π_R , and Σ_η as follows:

1. Obtain the matrix C_{IRF} as the lower-triangular Cholesky factorization such that $\Sigma_\varepsilon = C_{IRF}C'_{IRF}$ (MATLAB: `C=chol(Sigmae,'lower')`)

2. Further decompose C_{IRF} into a lower triangular matrix with unit coefficients along the diagonal R and a diagonal matrix D_{IRF} , i.e. $C_{IRF} = RD_{IRF}$, where R has ones along the diagonal and D_{IRF} is a diagonal matrix. (MATLAB: $D=\text{diag}(\text{diag}(C)); R = C * \text{inv}(D)$)

3. We can then multiply (E.1) by R^{-1} to get

$$R^{-1}Y_{IRF,t} = R^{-1}\Pi Y_{IRF,t-1} + R^{-1}\varepsilon_t. \quad (\text{E.6})$$

The variance-covariance matrix of $\eta_t = R^{-1}\varepsilon_t$ is diagonal with

$$E \left[R^{-1}\varepsilon_t (R^{-1}\varepsilon_t)' \right], \quad (\text{E.7})$$

$$= R^{-1}\Sigma_\varepsilon R^{-1'}, \quad (\text{E.8})$$

$$= R^{-1}C_{IRF}C'_{IRF}R^{-1'}, \quad (\text{E.9})$$

$$= R^{-1}RD_{IRF}D'_{IRF}R'R^{-1'}, \quad (\text{E.10})$$

$$= D_{IRF}D'_{IRF}. \quad (\text{E.11})$$

We therefore define:

$$\eta_t = R^{-1}\varepsilon_t, \quad (\text{E.12})$$

$$\Sigma_\eta = D_{IRF}D'_{IRF}, \quad (\text{E.13})$$

$$\Pi_R = R^{-1}\Pi. \quad (\text{E.14})$$

We can now solve for impulse responses. For the output gap impulse responses we start with a unit standard deviation shock to the output gap. We therefore look at macroeconomic impulse responses, where the period 1 shock equals

$$\eta_1 = [\sigma(\eta_1), 0, 0]', \quad (\text{E.15})$$

and shocks in all other periods equal zero. Equivalently, we look at the impulse response to the shock

$$\varepsilon_1 = R[\sigma(\eta_1), 0, 0]', \quad (\text{E.16})$$

and $\varepsilon_t = 0 \forall t > 1$. The n response to a one standard deviation shock to the output gap then is computed as:

$$\Pi^{n-1}\varepsilon_1, \quad (\text{E.17})$$

$$= \Pi^{n-1}R[\sigma(\eta_1), 0, 0]'. \quad (\text{E.18})$$

Impulse responses to inflation and Federal Funds rate shocks are computed analogously as $\Pi^{n-1}R[0, \sigma(\eta_2), 0]'$ and $\Pi^{n-1}R[0, 0, \sigma(\eta_3)]'$.

E.2 Expressions for correlations with the output gap

The SMM estimation procedure matches the correlation between the 20-quarter average Federal Funds rate and the output gap. In the data, we compute this correlation as

$$\widehat{Corr}_{xi} \equiv \widehat{Corr} \left(\frac{1}{20} (i_t + i_{t+1} + \dots + i_{t+19}), x_t \right), \quad (\text{E.19})$$

where t ranges from the first to the last quarter in either period 1 or period 2. The empirical correlation of 5-year average inflation with the output gap is computed analogously, with $i_t, i_{t+1}, \dots, i_{t+19}$ replaced by $\pi_t, \pi_{t+1}, \dots, \pi_{t+19}$. The empirical inflation-output gap correlation is computed as

$$\widehat{Corr} (\pi_t, x_t), \quad (\text{E.20})$$

where t ranges from the first to the last quarter in either period 1 or period 2.

The model correlation of the 5-year average Federal Funds rate with the output gap is computed analytically according to:

$$Corr_{t-1} \left(\frac{1}{20} (i_t + i_{t+1} + \dots + i_{t+19}), x_t \right), \quad (\text{E.21})$$

$$= \frac{Q_M \Sigma_u (e_3 (I - P)^{-1} (I - P^{20})Q + 20e_3)'}{\sqrt{Q_M \Sigma_u Q_M'} \sqrt{(e_2 (I - P)^{-1} (I - P^{20})Q + 20e_3) \Sigma_u (e_3 (I - P)^{-1} (I - P^{20})Q + 20e_3)'}}. \quad (\text{E.22})$$

The model correlation of 5-year average inflation with the output gap equals:

$$Corr_{t-1} \left(\frac{1}{20} (\pi_t + \pi_{t+1} + \dots + \pi_{t+19}), x_t \right), \quad (\text{E.23})$$

$$= \frac{Q_M \Sigma_u (e_2 (I - P)^{-1} (I - P^{20})Q + 20e_3)'}{\sqrt{Q_M \Sigma_u Q_M'} \sqrt{(e_2 (I - P)^{-1} (I - P^{20})Q + 20e_3) \Sigma_u (e_2 (I - P)^{-1} (I - P^{20})Q + 20e_3)'}}. \quad (\text{E.24})$$

The model correlation of one-quarter inflation with the output gap equals:

$$Corr_{t-1} (\pi_t, x_t) = \frac{Q_M \Sigma_u (e_2 Q + e_3)'}{\sqrt{Q_M \Sigma_u Q_M'} \sqrt{(e_2 Q + e_3) \Sigma_u (e_2 Q + e_3)'}} \quad (\text{E.25})$$

E.3 Asset return loadings on impulse responses

We compute model asset return loadings on impulse responses as follows.

1. We use the model to generate a simulated sample for $x_t^{sim}, \pi_t^{sim}, i_t^{sim}$ and bond and stock return components $xr_t^{sim,\delta}, xr_t^{sim,cf,\delta}, xr_t^{sim,rr,\delta}, xr_t^{sim,rp,\delta}, xr_{n,t}^{sim,\$}, xr_{n,t}^{sim,cf,\$}, xr_{n,t}^{sim,rr,\$}, xr_{n,t}^{sim,rp,\$}$. The simulated sample has the same length as our data.
2. We estimate the VAR(1) (E.1) on the simulated model data, giving simulated VAR(1) residuals ε_t^{sim} . We use the methodology in Section E.1 to obtain orthogonal innovations $\eta_t^{sim} = [\eta_{x,t}^{sim}, \eta_{\pi,t}^{sim}, \eta_{i,t}^{sim}]$.
3. We regress $xr_t^{sim,\delta}, xr_t^{sim,cf,\delta}, xr_t^{sim,rr,\delta}, xr_t^{sim,rp,\delta}, xr_{n,t}^{sim,\$}, xr_{n,t}^{sim,cf,\$}, xr_{n,t}^{sim,rr,\$}, xr_{n,t}^{sim,rp,\$}$ onto $\eta_{x,t+1}^{sim}, \eta_{\pi,t}^{sim}$, and $\eta_{i,t}^{sim}$. We regress onto $\eta_{x,t+1}^{sim}$ rather than $\eta_{x,t}^{sim}$ to account for the fact that $\eta_{x,t+1}^{sim}$ reflects news about x_t^{sim} , because $Y_{IRF,t}^{sim} = [x_{t-1}^{sim}, \pi_t^{sim}, i_t^{sim}]$.
4. We standardize the estimated slope coefficients by the standard deviations of $\eta_{x,t+1}^{sim}, \eta_{\pi,t}^{sim}$, and $\eta_{i,t}^{sim}$.
5. We report standardized slope coefficients averaged across 100 independent model simulations

E.4 Confidence intervals and objective function

We use a bootstrap method to compute confidence intervals for the empirical impulse responses shown in Figures 3 through 5 and for the variances of the impulse responses used in the SMM estimation. Let Π and Σ_ε denote the VAR(1) coefficient matrix and the variance-covariance matrix of shocks from estimating (E.1) on actual data. We then generate bootstrapped data from this VAR(1) by simulating $Y_{IRF,t}^{boot}$ of identical sample length as the true data according to

$$Y_{IRF,t}^{boot} = \hat{\Pi} Y_{IRF,t-1}^{boot} + \varepsilon_t^{boot}, \quad (\text{E.26})$$

where ε_t^{boot} are drawn as iid normal with mean zero and variance-covariance Σ_ε . On the bootstrapped data, we then apply the methodology for IRFs described in Section E.1. That is, we re-estimate (E.1) on the bootstrapped data and use the resulting estimates to construct bootstrapped impulse response functions. We generate 1000 independent bootstrap samples. Figures 3 through 5 show confidence intervals, such that 95% of the time the bootstrapped impulse responses are within the interval.

For our objective function, we define the empirical target moments as follows. $\hat{\Psi}$ is $[52 \times 1]$. It includes $51 = 6 \cdot 9 - 3$ impulse responses and the 5-year average Federal Funds rate-output gap correlation. We have 51 impulse response moments, because we have nine impulse responses at one (shock period), two, four, 12, 20, and 40 quarters each. However, three of the shock period impulse responses are zero by our choice of orthogonalization and we exclude them from the objective function. The last element of $\hat{\Psi} - \Psi(param)$ is the square-root of the 20-quarter Federal Funds

rate-output gap correlation in the model minus the data, defined as

$$\sqrt{\left| \widehat{Corr}_{t-1}((i_t + i + t + 1 + \dots + i_{19}), x_t) - \widehat{Corr}\left(\frac{1}{20}(i_t + i + t + 1 + \dots + i_{19}), x_t\right) \right|}.$$

Let \hat{V} denote the bootstrapped variance-covariance matrix of $\hat{\Psi}^{boot} - \hat{\Psi}$, where the 51 first elements of $\hat{\Psi}^{boot} - \hat{\Psi}$ are the difference between the bootstrapped impulse response moments and the data impulse responses and the last element is $\sqrt{\widehat{Corr}_{xi}^{boot} - \widehat{Corr}_{xi}}$ and \widehat{Corr}_{xi} and $\widehat{Corr}_{xi}^{boot}$ denote the 5-year average Federal Funds rate-output gap correlation computed according to (E.19) in actual and bootstrapped data. We then define the weighting matrix \hat{W} for the SMM objective function as the diagonal matrix with the inverse variances for the 51 impulse response moments and 200 along the diagonal:

$$\hat{W} = \text{diag}(\text{inv}(\hat{V}_{1,1}), \text{inv}(\hat{V}_{2,2}), \dots, \text{inv}(\hat{V}_{51,51}), 200). \quad (\text{E.27})$$

E.5 Grid search

We minimize the SMM objective function by grid search. Solving for the macroeconomic dynamics once and simulating the macroeconomic impulse response functions (but not solving for asset prices) takes about 1.2 seconds. So the reason that the estimation is computationally intensive is that we have a high-dimensional parameter space, combined with the need to simulate impulse responses to adjust for small sample effects in plausibly short empirical samples. We are minimizing over 12 parameters, so we have a high-dimensional optimization problem, for which gradient-based optimization methods do not work well. We solve the challenges posed by high-dimensional optimization by a) dividing parameters into blocks b) using the Harvard Odyssey computing cluster c) grid search along an appropriate grid.

We minimize $Obj(params)$ using a two-step grid search. To reduce the dimensionality of the grid search, we separate the parameters into those determining the slope coefficients in the VAR(1) $params_{slope} = [p_{\pi i}, p_{\pi \pi}, p_{\pi i}, p_{ix}, p_{i\pi}, p_{ii}]$ and those determining the shock variances and correlations $params_{shocks} = [\sigma_{\pi}, \sigma_i, \sigma_*, \rho_{\pi i}, \rho_{\pi*}, \rho_{i*}]$. The grid search first minimizes over $params_{slope}$ while initializing the shock parameters at $params_{vol} = [0.1, 0.1, 0.1, 0, 0, 0]$. This first grid search step solves and simulates macroeconomic dynamics over a grid. The grid for this first step consists of 10 equally-spaced points between -1 and 1 for every parameter in $params_{slope}$. In this first step, we thus solve and simulate the macroeconomic dynamics 10^6 times. We achieve this high number of model evaluations by running up to 1000 model simulations in parallel on the Harvard Odyssey computing cluster. In a second step, we minimize the objective function over $params_{vol}$, while holding constant $params_{slope}$ at the previously estimated values. We use a grid with 10 points for each of the 6 parameters in $params_{slope}$. For the volatilities, we use equal-spaced grids from 0.01% to 0.3%. For the correlations, we use equal-spaced grids from -0.99 to 0.99 .

E.6 Standard errors and hypothesis tests

E.6.1 Standard errors

This Section provides details on the parameter standard errors in Table 1. Despite our non-standard weighting matrix, we can compute standard errors using standard methods, similarly to Bekaert and Engstrom (2017) and CEE. Under the usual conditions, the asymptotic distribution of the parameter estimates \widehat{params} around their true values $params$ is given by:

$$(\widehat{params} - params) \sim N\left(0, \hat{V}_{params}\right), \quad (\text{E.28})$$

$$\hat{V}_{params} = \left(\hat{M}_{SE}^{-1} \hat{H}'_{SE} \hat{W} \hat{V} \hat{W} \hat{H}_{SE} \hat{M}_{SE}^{-1}\right) \quad (\text{E.29})$$

\hat{W} is the diagonal weighting matrix used in the SMM objective function. \hat{V} is the variance-covariance matrix of empirical moments. Both \hat{V} and \hat{W} are obtained as described in Section E.4. The matrix $\hat{H}_{SE} = \nabla \Psi(\widehat{params})$ is the Jacobian of the model moments evaluated at the vector of point estimates, \widehat{params} . Finally, $\hat{M}_{SE} \equiv \hat{H}'_{SE} \hat{W} \hat{H}_{SE}$.

We obtain the i 'th column of \hat{H}_{SE} as the numerical derivative of the model moments with respect to parameter i . Specifically, we simulate moments while moving parameter i by $\pm eps_i$ from its point estimate, and dividing the difference by $2eps_i$. We then compute \hat{H}_{SE} according to:

$$\hat{H}_{SE} = \left[\frac{\Psi(\widehat{params} + e_1 \epsilon) - \Psi(\widehat{params} - e_1 \epsilon)}{2eps_1}, \dots, \frac{\Psi(\widehat{params} + e_{12} \epsilon) - \Psi(\widehat{params} - e_{12} \epsilon)}{2eps_{12}} \right]. \quad (\text{E.30})$$

The matrix \hat{H}_{SE} is $[52 \times 12]$, because we have 52 moments and 12 parameters.

There are two practical challenges when computing standard errors. The first one is that the 12×12 matrix \hat{M}_{SE} may be hard to invert if its eigenvalues have drastically different orders of magnitude. We deal with this first challenge by scaling the standard deviations by 400 before doing the standard errors calculation. The second challenge is that our moments are simulated, so there is simulation noise around the true $\Psi(\widehat{params} \pm e_i \epsilon)$. We deal with this second challenge by choosing values for eps_i that are large enough that movements in model moments are not dominated by simulation noise. We allow for different epsilons eps_i for different parameters, because parameters have different scales. Specifically, we set $\epsilon = 0.50$ for the lag and correlation parameters (which are in natural units) and $\epsilon = 0.10$ for the standard deviations of shocks (which are already scaled by 400 to make them of comparable magnitude to the lag parameters).

E.6.2 Hypothesis tests

Let \widehat{params}_1 and \widehat{params}_2 denote the parameter point estimates for periods 1 and 2. Let $\hat{V}_{params,1}$ and $\hat{V}_{params,2}$ denote the asymptotic variance-covariance matrices for periods 1 and 2, as defined in equation (E.29).

Table 1 reports the results from testing whether parameter i is equal in periods 1 and 2. Under the null hypothesis that $params_1(i) = params_2(i)$, we have that:

$$\widehat{params}_2(i) - \widehat{params}_1(i) \sim N\left(0, e_i \left(V_{params,1} + \hat{V}_{params,2}\right) e_i'\right). \quad (\text{E.31})$$

The test statistic is:

$$\left(\widehat{params}_2(i) - \widehat{params}_1(i)\right) \left(e_i \left(\hat{V}_{params,1} + \hat{V}_{params,2}\right) e_i'\right)^{-1} \left(\widehat{params}_2(i) - \widehat{params}_1(i)\right) \sim \chi_1^2. \quad (\text{E.32})$$

We compute p-value for the test that $params_1(i) = params_2(i)$ by plugging the test statistic (E.32) into the CDF of a chi-squared distribution with 1 degree of freedom.

E.7 Kalman filter for Figure 1

In this section, we use subscript $_{kal}$ to denote parameters and processes specific to the Kalman filter to distinguish them from variables with similar names elsewhere in the paper. Let β_t denote the bond beta estimated from daily returns over the past quarter t . We model β_t as an unobserved AR(1) component plus white measurement noise. We denote the deviation of β_t from its mean μ_{kal} by:

$$y_{kal,t} = \beta_t - \mu_{kal}. \quad (\text{E.33})$$

We assume that $y_{kal,t}$ satisfies the plant and observation equations:

$$x_{kal,t} = A_{kal}x_{kal,t-1} + u_{kal,t}, \quad (\text{E.34})$$

$$y_{kal,t} = x_{kal,t} + v_{kal,t}, \quad (\text{E.35})$$

where $u_{kal,t}$ and $v_{kal,t}$ are mean zero, independent random variables with variances $\sigma_{kal,u}^2$ and $\sigma_{kal,v}^2$. The unobserved component $x_{kal,t}$ is the latent de-meaned beta of nominal bonds, that we want to estimate.

We can re-write the system (E.34) and (E.35) as:

$$x_{kal,t} = A_{kal}x_{kal,t-1} + \epsilon_{kal,t}, \quad (\text{E.36})$$

$$y_{kal,t} = C_{kal}x_{kal,t-1} + \eta_{kal,t}, \quad (\text{E.37})$$

where

$$C_{kal} = A_{kal}, \quad (\text{E.38})$$

$$N_{kal} = \text{Var}(\epsilon_{kal,t}) = \sigma_{kal,u}^2, \quad (\text{E.39})$$

$$L_{kal} = \text{Cov}(\epsilon_{kal,t}, \eta_{kal,t}) = \sigma_{kal,u}^2, \quad (\text{E.40})$$

$$M_{kal} = \text{Var}(\eta_{kal,t}) = \sigma_{kal,u}^2 + \sigma_{kal,v}^2. \quad (\text{E.41})$$

Let $W_{kal,t}$ denote the information available up to time t from observing $\dots, \beta_{t-1}, \beta_t$. Suppose that conditional on $W_{kal,0}$ the initial state $x_{kal,0}$ is distributed $N(\hat{x}_{kal,0}, V_{kal,0})$ and the state and observations obey the recursions (E.36) through (E.37). Standard Kalman filter results then imply that conditional on $W_{kal,t}$, the current state is distributed $N(\hat{x}_{kal,t}, V_{kal,t})$ and that the conditional mean and variance obey the updating recursions:²

$$\hat{x}_{kal,t} = A_{kal}\hat{x}_{kal,t-1} + H_{kal,t}(y_{kal,t} - C_{kal}\hat{x}_{kal,t-1}), \quad (\text{E.42})$$

$$\begin{aligned} V_{kal,t} &= N_{kal} + A_{kal}V_{kal,t-1}A'_{kal} \\ &\quad - (L_{kal} + A_{kal}V_{kal,t-1}C'_{kal}) \cdot \\ &\quad (M_{kal} + C_{kal}V_{kal,t-1}C'_{kal})^{-1} (L'_{kal} + C_{kal}V_{kal,t-1}A'_{kal}), \end{aligned} \quad (\text{E.43})$$

$$H_{kal,t} = (L_{kal} + A_{kal}V_{kal,t-1}C'_{kal})(M_{kal} + C_{kal}V_{kal,t-1}C'_{kal})^{-1}. \quad (\text{E.44})$$

The distribution of $y_{kal,t}$ conditional on the information available at time $t-1$ is normal with conditional mean:

$$E(y_{kal,t}|W_{kal,t-1}) = C_{kal}\hat{x}_{kal,t-1}, \quad (\text{E.45})$$

and variance

$$\text{Var}(y_{kal,t}|W_{kal,t-1}) = M_{kal} + C_{kal}V_{kal,t-1}C'_{kal}. \quad (\text{E.46})$$

The log-likelihood hence obeys the recursion:

$$\begin{aligned} LL(y_{kal,1}, y_{kal,2}, \dots, y_{kal,t}) &= LL(y_{kal,1}, y_{kal,2}, \dots, y_{kal,t-1}) \\ &\quad - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log(M_{kal} + C_{kal}V_{kal,t-1}C'_{kal}) \\ &\quad - \frac{(y_{kal,t} - C_{kal}\hat{x}_{kal,t-1})^2}{2(M_{kal} + C_{kal}V_{kal,t-1}C'_{kal})}. \end{aligned} \quad (\text{E.47})$$

We estimate the parameters μ_{kal} , A_{kal} , $\sigma_{kal,u}$, $\sigma_{kal,v}$ by maximizing the likelihood function (E.47).

²We substitute into the Kalman filter updating equations from Richard Weber's Optimization and Control class notes available at <http://www.statslab.cam.ac.uk/~rrw1/oc/index.html>.

We consider a 95% confidence interval for the unobserved bond beta $x_{kal,t} + \mu_{kal}$ conditional on $W_{kal,t}$:

$$CI(x_{kal,t}|W_{kal,t}) = \left[\hat{x}_{kal,t} + \mu_{kal} - 1.96\sqrt{V_{kal,t}}, \hat{x}_{kal,t} + \mu_{kal} + 1.96\sqrt{V_{kal,t}} \right]. \quad (\text{E.48})$$

The estimated parameters for the betas are as follows: $\mu_{kal} = 0.60$, $A_{kal} = 0.96$, $\sigma_{kalmna,u} = 0.04$, $\sigma_{kal} = 0.08$. We run exactly the same Kalman filter for the quarter-end bond-stock correlations of daily returns and obtain: $\mu_{kal,corr} = 0.08$, $A_{kal,corr} = 0.95$, $\sigma_{kal,corr,u} = 0.10$, $\sigma_{kal,corr,v} = 0.17$. These parameter estimates show that both betas and correlations are estimated to have substantial persistence, but also a high volatility of iid noise, so filtering is useful in filtering out the noise.

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